

Business Calculus Final Examination

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1. Use the linear approximation to approximate the value of $\sqrt[4]{8.998}$. Compare your result with the exact value.

$$f(x) = \sqrt[4]{x+8} \quad x = 8$$

$$f'(x) = -\frac{1}{4(x+8)^{3/4}}$$

$$f'(8) = \frac{-1}{4(8)} = -\frac{1}{32}$$

$$f(8) = 2$$

$$y = 2 - \frac{1}{32}(x-8)$$

$$= -\frac{1}{32}x + \frac{1}{4} + 2$$

$$= -\frac{1}{32}x + \frac{9}{4}$$

$$\sqrt[4]{8.998} = -\frac{1}{32}(8.998) + \frac{9}{4} = 1.9688125$$

$$\sqrt[4]{8.998} = 1.73195$$

2. A box whose base length is 4 times the base width is constructed to enclose 32 in³. The cost of the material of the sides is \$2/in² and the cost of the top and bottom is \$6/in². Determine the dimensions of the box that minimizes the cost.

$$x, 4x, y$$

$$x(4x)y = 32$$

$$y = \frac{32}{4x^2} = \frac{8}{x^2}$$

$$\begin{aligned} \text{Cost} &= 4x^2(2)(6) + (2xy + 2(4xy))2 \\ &= 48x^2 + 20xy \\ &= 48x^2 + 20x \cdot \frac{8}{x^2} = 48x^2 + \frac{160}{x} \end{aligned}$$

$$C'(x) = 96x - \frac{160}{x^2} = 0$$

$$x = \left(\frac{160}{96}\right)^{1/3} \approx 1.1856$$

$$y = \frac{8}{1.1856^2} = 5.6913$$

Dimensions are

1.19 inches, 4.76 in, 5.69 in

3. Use the definition of the derivative to find $f'(x)$ where $f(x) = 3 - \sqrt{1-x}$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{3 - \sqrt{1-x-h} - (3 - \sqrt{1-x})}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{1-x} - \sqrt{1-x-h}}{h} \left(\frac{\sqrt{1-x} + \sqrt{1-x-h}}{\sqrt{1-x} + \sqrt{1-x-h}} \right)$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{1-x} - (\cancel{1-x} - h)}{h(\sqrt{1-x} + \sqrt{1-x-h})}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1-x} + \sqrt{1-x-h})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{1-x} + \sqrt{1-x-h}}$$

$$= \frac{1}{2\sqrt{1-x}}$$

4. Find $f'(x)$ (do not simplify):

(a) $f(x) = (4^x - x) \cdot \left(\frac{1}{3+3^{2x}}\right)$

(b) $f(x) = \sqrt{\ln\left(\frac{2+x^2}{x}\right)}$

$$\text{a) } f'(x) = (4^x - x) \left(\frac{-1}{(3 + 3^{2x})^2} \right) \cdot (2 \ln 3 (3^{2x})) \\ + \frac{1}{3 + 3^{2x}} (4^x \ln 4 - 1)$$

$$\text{b) } f'(x) = \frac{1}{2 \sqrt{\ln\left(\frac{2+x^2}{x}\right)}} \left[\frac{2x}{2+x^2} - \frac{1}{x} \right]$$

5. Let

$$f(x) = \begin{cases} -x + 2 & x \leq 0, \\ 2x^2 + 2 & 0 < x \leq 2, \\ x^3 + 3 & x > 2. \end{cases}$$

Determine where $f(x)$ is continuous.

$$\underline{x=0} \quad \lim_{x \rightarrow 0^-} f(x) = 2$$

$$\lim_{x \rightarrow 0^+} f(x) = 2$$

$$\lim_{x \rightarrow 0} f(x) = 2, \quad f(0) = 2$$

Continuous at $x=0$

$$\underline{x=2} \quad \lim_{x \rightarrow 2^-} f(x) = 10$$

$$\lim_{x \rightarrow 2^+} f(x) = 11$$

$$\lim_{x \rightarrow 2} f(x) \quad \text{DNE}$$

Not Continuous at $x=2$

$$(-\infty, 2) \cup (2, \infty)$$

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6. Calculate the indefinite integrals:

$$\int (x^4 - \sqrt{x} + 10) dx$$

and

$$\int 3x^3 \sqrt{1-x^4} dx.$$

$$\int (x^4 - \sqrt{x} + 10) dx = \frac{x^5}{5} - \frac{2}{3} x^{3/2} + 10x + C$$

$$\int 3x^3 \sqrt{1-x^4} dx$$

$$u = 1 - x^4$$
$$du = -4x^3 dx$$

$$= \int -\frac{3}{4} \sqrt{u} du$$

$$= -\frac{3}{4} \cdot \frac{2}{3} u^{3/2} + C$$

$$= -\frac{1}{2} (1 - x^4)^{3/2} + C$$

7. The half-life of radium-226 is about 1600 years. A sample of radium-226 has a mass of 250 mg. What is the mass of the sample that remains after 300 years?

$$r = \frac{\ln 0.5}{1600}$$

$$A = 250 e^{\frac{\ln 0.5}{1600} t}$$

$$A(300) = 250 e^{\frac{\ln 0.5}{1600} (300)}$$

$$= 219.53 \text{ mg}$$

8. Let

$$f(x) = x - \frac{1}{x^2}, \quad x \neq 0$$

(1) Find the local maximum and minimum values of $f(x)$ by using the second derivative test.

(2) Find the absolute maximum and minimum values of $f(x)$ on $(-\infty, \infty)$.

$$f'(x) = 1 + \frac{2}{x^3} = 0$$

$$\frac{x^3 + 2}{x^3} = 0$$

$$x^3 = -2$$

$$x = \sqrt[3]{-2}$$

$\left(-\infty, \sqrt[3]{-2}\right)$	$\left(\sqrt[3]{-2}, 0\right)$	$\left(0, \infty\right)$
$f' > 0$	$f' < 0$	$f' > 0$

increasing

decreasing

increasing

a) Local max $\left(\sqrt[3]{-2}, \sqrt[3]{-2} - \frac{1}{\frac{2}{3}}\right)$
 No Local min

b) No absolute maximum or minimum

9. Find the area of the region bounded by

$$y = -x^2 + 4 \text{ and } y = 2x - 1.$$

$$-x^2 + 4 = 2x - 1$$

$$x^2 + 2x - 5 = 0 \quad x = -1 \pm \sqrt{6}$$

$$\text{Area} = \int_{-1-\sqrt{6}}^{-1+\sqrt{6}} (-x^2 + 4) - (2x - 1) dx$$

$$= \int_{-1-\sqrt{6}}^{-1+\sqrt{6}} (-x^2 - 2x + 5) dx$$

$$= \left[\frac{-x^3}{3} - x^2 + 5x \right]_{-1-\sqrt{6}}^{-1+\sqrt{6}}$$

$$= - \frac{(-1+\sqrt{6})^3 - (-1-\sqrt{6})^3}{3} + 14\sqrt{6}$$

$$= -6\sqrt{6} + 14\sqrt{6}$$

$$= 8\sqrt{6}$$

10. Find the domain of

$$f(x) = \frac{1}{\sqrt{x-1} - \sqrt{2-x}}.$$

$$\sqrt{x-1} - \sqrt{2-x} \neq 0$$

$$\left(\sqrt{x-1}\right)^2 = \left(\sqrt{2-x}\right)^2$$

$$x-1 = 2-x$$

$$2x = 3$$

$$x = 3/2$$

$$x-1 \geq 0 \quad x \geq 1$$

$$2-x \geq 0 \quad x \leq 2$$

$$x \geq 1 \text{ and } x \leq 2 \text{ and } x \neq 3/2$$

$$\left[1, \frac{3}{2}\right) \cup \left(\frac{3}{2}, 2\right]$$